3 Applications of Differential Equations

Differential equations are absolutely fundamental to modern science and engineering. Almost all of the known laws of physics and chemistry are actually differential equations, and differential equation models are used extensively in biology to study biochemical reactions, population dynamics, organism growth, and the spread of diseases.

A **mathematical model** is a description of a real-world system using mathematical language and ideas.

The most common use of differential equations in science is to model **dynamical systems**, i.e. systems that change in time according to some fixed rule. For such a system, the independent variable is t (for time) instead of x, meaning that equations are written like

$$\frac{dy}{dt} = t^3 y^2 \qquad \text{instead of} \qquad y' = x^3 y^2.$$

In addition, the letter y is usually replaced by a letter that represents the variable under consideration, e.g. M for mass, P for population, T for temperature, and so forth.

Exponential Growth and Decay

Perhaps the most common differential equation in the sciences is the following.

THE NATURAL GROWTH EQUATION

The **natural growth equation** is the differential equation

$$\frac{dy}{dt} = ky$$

where k is a constant. Its solutions have the form

$$y = y_0 e^{kt}$$

where $y_0 = y(0)$ is the initial value of y.

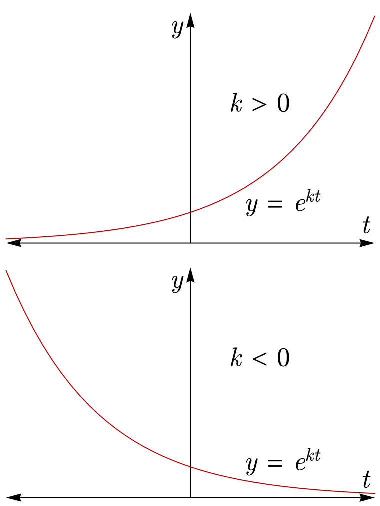

Figure 1: Exponential growth and decay.

The constant k is called the **rate constant** or **growth constant**, and has units of inverse time (number per second). The sign of k governs the behavior of the solutions:

- If $k > 0$, then the variable y increases exponentially over time. This is called **exponential growth**.
- If $k < 0$, then the variable y decreases over time, approaching zero asymptotically. This is called **exponential decay**.

See Figure 1 for sample graphs of $y = e^{kt}$ in these two cases. In the case where k is negative, the natural growth equation can also be written

$$\frac{dy}{dt} = -ry$$

where $r = |k|$ is positive, in which case the solutions have the form $y = y_0 e^{-rt}$.

The following examples illustrate several instances in science where exponential growth or decay is relevant.

EXAMPLE 1 Consider a colony of bacteria in a resource-rich environment. Here "resource-rich" means, for example, that there is plenty of food, as well as space for the colony to grow. In such an environment, the population P of the colony will grow, as individual bacteria reproduce via binary fission.

Assuming that no bacteria die, the rate at which such a population grows will be proportional to the number of bacteria. For example, the population might increase at a rate of 5% per minute, regardless of its size. Intuitively, this is because the rate at which individual bacterial cells divide does not depend on the number of cells.

We can express this rule as a differential equation:

$$\frac{dP}{dt} = kP.$$

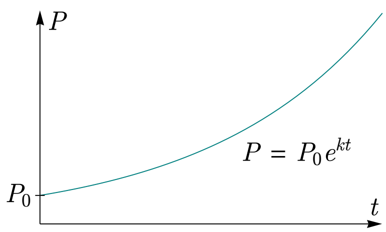

Figure 2: Exponential growth of a bacteria population.

Here k is a constant of proportionality, which can be interpreted as the rate at which the bacteria reproduce. For example, if $k = 3$/hour, it means that each individual bacteria cell has an average of 3 offspring per hour (not counting grandchildren).

It follows that the population of bacteria will grow exponentially with time:

$$P = P_0 e^{kt}$$

where P_0 is the population at time $t = 0$ (see Figure 2). ■

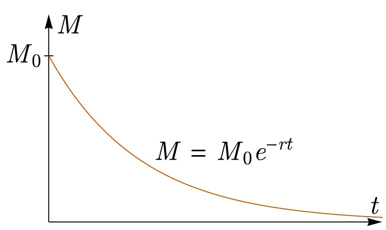

Figure 3: Exponential decay of a radioactive isotope.

EXAMPLE 2 Consider a sample of a certain radioactive isotope. The atoms of such an isotope are unstable, with a certain proportion decaying each second. In particular, the mass M of the sample will decrease as atoms are lost, with the rate of decrease proportional to the number of atoms. We can write this as a differential equation

$$\frac{dM}{dt} = -rM,$$

where r is a constant of proportionality. It follows that the mass of the sample will decay exponentially with time:

$$M = M_0 e^{-rt},$$

where M_0 is the mass of the sample at time $t = 0$ (see Figure 2). ■

One important measure of the rate of exponential decay is the **half life**. Given a decaying variable

$$y = y_0 e^{-rt} \qquad (r > 0)$$

the half life is the amount of time that it takes for y to decrease to half of its original value. The half life can be obtained by substituting $y = y0/2$

$$\frac{y_0}{2} = y_0 e^{-rt}$$

and then solving for t.

Similarly, given a growing variable

$$y = y_0 e^{kt} \qquad (k > 0)$$

we can measure the rate of exponential growth using the **doubling time**, i.e. the amount of time that it takes for y to grow to twice its original value. The doubling time can be obtained by substituting $y = 2y_0$ and then solving for t.

The following example illustrates a more complicated situation where the natural growth equation arises.

EXAMPLE 3 Figure 4 shows a simple kind of electric circuit known as an **RC circuit**. This circuit has two components:

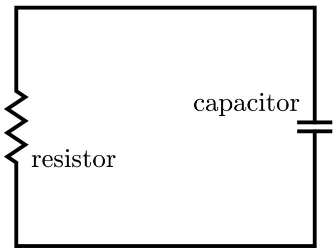

Figure 4: An RC circuit.

- A **resistor** is any circuit component—such as a light bulb—that resists the flow of electric charge. Resistors obey **Ohm's law**

$$V = IR,$$

where V is the voltage applied to the resistor, I is the rate at which charge flows through the resistor, and R is a constant called the **resistance**.

- A **capacitor** is a circuit component that stores a supply of electric charge. When it is attached to a resistor, the capacitor will push this charge through the resistor, creating electric current. Capacitors obey the equation

$$V = \frac{Q}{C},$$

where Q is the charge stored in the capacitor, C is a constant called the **capacitance** of the capacitor, and V is the resulting voltage.

In an RC circuit, the voltage produced by a capacitor is applied directly across a resistor. Setting the two formulas for V equal to each other gives

$$IR = \frac{Q}{C}.$$

Moreover, the rate I at which charge flows through the resistor is the same as the rate at which charge flows out of the capacitor, so

$$I = -\frac{dQ}{dt}.$$

Putting these together gives the differential equation

$$\left(-\frac{dQ}{dt}\right)R = \frac{Q}{C},$$

or equivalently

$$\frac{dQ}{dt} = -\frac{1}{RC}Q.$$

It follows that the amount of charge held in the capacitor will decay exponentially over time

$$Q = Q_0 e^{-rt}$$

where $r = 1/(RC)$. In the case where the resistor is a light bulb, this means that the bulb will become dimmer and dimmer over time, although it will never quite go out. ■

Although the light bulb will technically never go out, in reality the light will become too faint to see after a short time.

Separation of Variables

Many differential equations in science are separable, which makes it easy to find a solution.

EXAMPLE 4 **Newton's law of cooling** is a differential equation that predicts the cooling of a warm body placed in a cold environment. According to the law, the rate at which the temperature of the body decreases is proportional to the difference of temperature between the body and its environment. In symbols

$$\frac{dT}{dt} = -k(T - T_e),$$

where T is the temperature of the object, T_e is the (constant) temperature of the environment, and k is a constant of proportionality.

We can solve this differential equation using separation of variables. We get

$$\int \frac{dT}{T - T_e} = \int -k\,dt,$$

so

$$\ln|T - T_e| = -kt + C.$$

Solving for T gives an equation of the form

$$T = T_e + Ce^{-kt}$$

where the value of C changed. This function decreases exponentially, but approaches T_e as $t \to \infty$ instead of zero (see Figure 5). ■

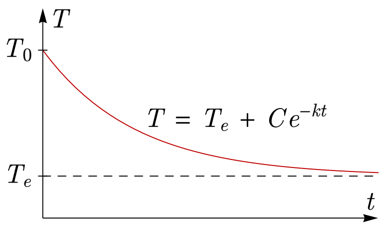

Figure 5: Cooling of a warm body.

EXAMPLE 5 In chemistry, the rate at which a given chemical reaction occurs is often determined by a differential equation. For example, consider the decomposition of nitrogen dioxide:

$$2\,NO_2 \longrightarrow 2\,NO + O_2.$$

Because this reaction requires two molecules of NO_2, the rate at which the reaction occurs is proportional to the *square* of the concentration of NO_2. That is,

$$\frac{d[NO_2]}{dt} = -k[NO_2]^2$$

where $[NO_2]$ is the concentration of NO_2, and k is a constant.

We are using the usual chemistry notation, where $[NO_2]$ denotes the concentration of NO_2. An alternative would be to use a single letter for this concentration, such as N.

We can solve this equation using separation of variables. We get

$$\int [NO_2]^{-2} d[NO_2] = \int -k\,dt$$

so

$$-[NO_2]^{-1} = -kt + C.$$

Solving for $[NO_2]$ gives

$$[NO_2] = \frac{1}{kt + C}$$

where the value of C changed. An example graph corresponding to this formula is shown in Figure 6. Unlike exponential decay, the concentration decreases very quickly at first, but then very slowly afterwards. ■

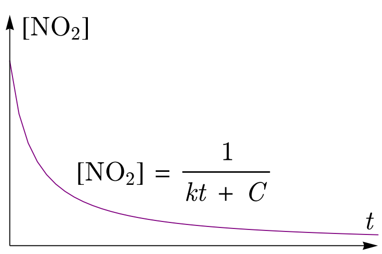

Figure 6: Decomposition of NO_2.

EXAMPLE 6 Consider a colony of bacteria growing in an environment with limited resources. For example, there may be a scarcity of food, or space constraints on the size of the colony. In this case, it is not reasonable to expect the colony to grow exponentially—indeed, the colony will unable to grow larger than some maximum population $P_{\max}$.

In this case, a common model for the growth of the colony is the **logistic equation**

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{P_{\max}}\right)$$

The maximum population $P_{\max}$ is called the **carrying capacity** of the bacteria colony in the given environment.

Here the factor of $1 - P/P_{\max}$ is unimportant when P is small, but when P is close to $P_{\max}$ this factor decreases the rate of growth. Indeed, in the case where $P = P_{\max}$, this factor forces dP/dt to be zero, meaning that the colony does not grow at all.

We can solve this differential equation using separation of variables, though it is a bit difficult. We begin by multiplying through by P_{max}

$$P_{\text{max}}\frac{dP}{dt} \;=\; kP(P_{\text{max}} - P).$$

We can now separate to get

$$\int \frac{P_{\text{max}}}{P(P_{\text{max}} - P)}\,dP \;=\; \int k\,dt.$$

The integral on the left is difficult to evaluate. The secret is to express the fraction as the sum of two simpler fractions:

$$\frac{P_{\text{max}}}{P(P_{\text{max}} - P)} \;=\; \frac{1}{P} + \frac{1}{P_{\text{max}} - P}.$$

This is a simple example of the integration technique known as **partial fractions decomposition**.

Each of the simpler fractions can then be integrated easily. The result is

$$\ln|P| \,-\, \ln|P_{\text{max}} - P| \;=\; kt + C.$$

We can use a logarithm rule to combine the two terms on the left:

$$\ln\left|\frac{P}{P_{\text{max}} - P}\right| \;=\; kt + C$$

so

$$\frac{P}{P_{\text{max}} - P} \;=\; Ce^{kt}.$$

Solving for P gives

$$P \;=\; \frac{P_{\text{max}}}{1 + Ce^{-kt}}$$

where the value of C changed.

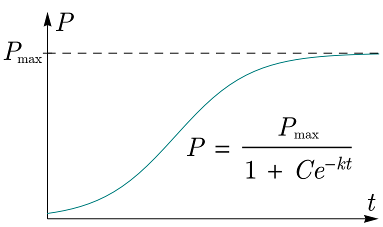

Figure 7: Logistic population growth.

Figure 7 shows the graph of a typical solution. Note that the population grows quickly at first, but the rate of increase slows as the population reaches the maximum. As $t \to \infty$, the population asymptotically approaches P_{max}. ■

In many of the examples we have seen, a differential equation includes an unknown constant k. This means that the general solution will involve *two* unknown constants (k and C). To solve such an equation, you will need two pieces of information, such as the values of $y(0)$ and $y'(0)$, or two different values of y.

The following example illustrates this procedure.

EXAMPLE 7 An apple pie with an initial temperature of 170 °C is removed from the oven and left to cool in a room with an air temperature of 20 °C. Given that the temperature of the pie initially decreases at a rate of 3.0 °C/min, how long will it take for the pie to cool to a temperature of 30 °C?

SOLUTION Assuming the pie obeys Newton's law of cooling (see Example 4), we have the following information:

$$\frac{dT}{dt} \;=\; -k(T - 20), \qquad T(0) = 170, \qquad T'(0) = -3.0,$$

where T is the temperature of the pie in celsius, t is the time in minutes, and k is an unknown constant.

We can easily find the value of k by plugging the information we know about $t = 0$ directly into the differential equation:

$$-2.5 \;=\; -k(170 - 20).$$

Second-Order Equations

Although first-order equations are the most common type in chemistry and biology, in physics most systems are modeled using second-order equations. This is because of Newton's second law:

$$F = ma.$$

The variable a on the right side of this equation is acceleration, which is the second derivative of position. Usually the force F depends on position as well as perhaps velocity, which means that Newton's second law is really a second-order differential equation.

For example, consider a mass hanging from a stretched spring. The force on such a mass is proportional to the position y, i.e.

$$F = -ky,$$

where k is a constant. Plugging this into Newton's second law gives the equation

$$-ky = my''.$$

The solutions to this differential equation involve sines and cosines, which is why a mass hanging from a spring will oscillate up and down. Similar differential equations can be used to model the motion of a pendulum, the vibrations of atoms in a covalent bond, and the oscillations of an electric circuit made from a capacitor and an inductor.

It follows that $k = 0.020/\text{sec}$. Now, the general solution to the differential equation is

$$T = 20 + Ce^{-kt}$$

and plugging in $t = 0$ gives

$$170 = 20 + C,$$

which means that $C = 150\,°\text{C}$. Thus

$$T = 20 + 150e^{-0.02t}.$$

To find how long it will take for the temperature to reach $30\,°\text{C}$, we plug in 30 for T and solve for t. The result is that $t = \boxed{135 \text{ minutes}}$. ■

The technique used in this example of substituting the initial conditions into the differential equation itself is quite common. It can be used whenever the differential equation itself involves an unknown constant, and we have information about both $y(0)$ and $y'(0)$.

EXERCISES

1. A sample of an unknown radioactive isotope initially weighs 5.00 g. One year later the mass has decreased to 4.27 g.

(a) How quickly is the mass of the isotope decreasing at that time?

(b) What is the half life of the isotope?

2. A cell culture is growing exponentially with a doubling time of 3.00 hours. If there are 5,000 cells initially, how long will it take for the cell culture to grow to 30,000 cells?

3. During a certain chemical reaction, the concentration of butyl chloride (C_4H_9Cl) obeys the rate equation

$$\frac{d[\mathrm{C_4H_9Cl}]}{dt} = -k[\mathrm{C_4H_9Cl}],$$

where $k = 0.1223/\text{sec}$. How long will it take for this reaction to consume 90% of the initial butyl chloride?

4. A capacitor with a capacitance of 5.0 coulombs/volt holds an initial charge of 350 coulombs. The capacitor is attached to a resistor with a resistance of 8.0 volt · sec/coulomb.

(a) How quickly will the charge held by the capacitor initially decrease?

(b) How quickly will the charge be decreasing after 20 seconds?

5. A bottle of water with an initial temperature of 25 °C is placed in a refrigerator with an internal temperature of 5 °C. Given that the temperature of the water is 20 °C ten minutes after it is placed in the refrigerator, what will the temperature of the water be after one hour?

6. In 1974, Stephen Hawking discovered that black holes emit a small amount of radiation, causing them to slowly evaporate over time. According to Hawking, the mass M of a black hole obeys the differential equation

$$\frac{dM}{dt} = -\frac{k}{M^2}$$

where $k = 1.26 \times 10^{23}\ \mathrm{kg}^3/\mathrm{year}$.

(a) Use separation of variables to find the general solution to this equation

(b) After a supernova, the remnant of a star collapses into a black hole with an initial mass of 6.00×10^{31} kg. How long will it take for this black hole to evaporate completely?

7. According to the **drag equation** the velocity of an object moving through a fluid can be modeled by the equation

$$\frac{dv}{dt} = -kv^2$$

where k is a constant.

(a) Find the general solution to this equation.

(b) An object moving through the water has an initial velocity of 40 m/s. Two seconds later, the velocity has decreased to 30 m/s. What will the velocity be after ten seconds?

8. A population of bacteria is undergoing logistic growth, with a maximum possible population of 100,000. Initially, the bacteria colony has 5,000 members, and the population is increasing at a rate of 400/minute.

(a) How large will the population be 30 minutes later?

(b) When will the population reach 80,000?

9. Water is being drained from a spout in the bottom of a cylindrical tank. According to **Torricelli's law**, the volume V of water left in the tank obeys the differential equation

$$\frac{dV}{dt} = -k\sqrt{V}$$

where k is a constant.

(a) Use separation of variables to find the general solution to this equation

(b) Suppose the tank initially holds 30.0 L of water, which initially drains at a rate of 1.80 L/min. How long will it take for tank to drain completely?

10. The **Gompertz equation** has been used to model the growth of malignant tumors. The equation states that

$$\frac{dP}{dt} = kP(\ln P_{\max} - \ln P)$$

where P is the population of cancer cells, and k and $P_{\max}$ are constants.

(a) Use separation of variables to find the general solution to this equation.

(b) A tumor with 5000 cells is initially growing at a rate of 200 cells/day. Assuming the maximum size of the tumor is $P_{\max} = 100{,}000$ cells, how large will the tumor be after 100 days?

Answers

1. (a) 0.67 g/yr (b) 4.39 years **2.** 7.75 hours **3.** 18.83 sec **4.** (a) 8.75 coulombs/sec (b) 5.3 coulombs/sec

5. 8.6°C **6.** (a) $M = \sqrt[3]{C - 3kt}$ (b) 5.71×10^{71} years **7.** (a) $v = \dfrac{1}{kt + C}$ (b) 15 m/s

8. (a) 39,697 (b) $t = 51.4$ min **9.** (a) $V = \dfrac{1}{4}(C - kt)^2$ (b) 33.3 min

10. (a) $P = P_{\max} \exp\left(Ce^{-kt}\right)$. (b) 45,468 cells