The Fourier Theory

A Technician Level Explanation

The Fourier Theory

The Fourier theory was discovered in 1826 by Jean Baptiste Fourier, a French physicist and mathematician. It is an infinite mathematical series that Fourier used in his analysis and design of heat and thermodynamic systems. It was later discovered that it could be used in almost any engineering analysis including mechanical and electrical/electronic systems.

The Fourier theory shows the relationship between the time and frequency domains. It provides a way to represent complex non-sinusoidal signals as the sum of harmonically related sine and/or cosine waves.

The Fourier theory states:

Any complex repetitive, non-sinusoidal waveform may be expressed mathematically as a fundamental sine wave at the signal frequency plus an infinite number of harmonic signals of varying amplitudes and phases.

Any repetitive waveform can be broken down into a series of sine waves at appropriate amplitudes and phases.

Harmonics

A harmonic is a sine wave whose frequency (f) is some integer (whole number) multiple of a fundamental sine wave frequency.

For example, a 2400 Hz sine wave is the 2nd harmonic of the fundamental 1200 Hz sine wave.

In another example, the second harmonic of a 4 MHz fundamental sine wave is 8 MHz. The 3rd harmonic is 12 MHz and so on. Expressed mathematically, the fundamental would be f_o, the 2nd harmonic would be 2 f_o, and the 7th harmonic 7 f_o.

Determining the Fundamental

A problem that occurs in the real world is determining the fundamental from the harmonic.

If the 5th harmonic is 75 MHz, what is the fundamental?

$75 = 5\ f_o$

$f_o = 75/5 = 15$ MHz

If the 3rd harmonic is 60 MHz, what is the fundamental?

$60 = 3\ f_o$

$f_o = 60/3 = 20$ MHz

Fourier Analysis

The Fourier theory is used to analyze complex periodic signals. By using a process based upon integral calculus, waveforms are reduced to a mathematical expression that is nothing more than the mathematical summation of the sine and/or cosine waves representing the fundamental and an infinite number of harmonics.

A general expression of the Fourier theory is:

$$y = A_o + \sum [A_n \sin 2\pi(nf)t + B_n \cos 2\pi(nf)t]$$

The term y is some signal that is a function of time or f(t). A_o is the DC component of the wave if it has one. The summation sign ($\sum$) indicates that a whole series of sine and cosine terms are to be added together. A is the sine amplitude, B is the cosine amplitude. The first term where n = 1 is the fundamental, the second term where n = 2 is the 2^{nd} harmonic, etc. The term n designates the higher integer values of the harmonics.

Fourier Expressions

Here is the previous Fourier expression expanded:

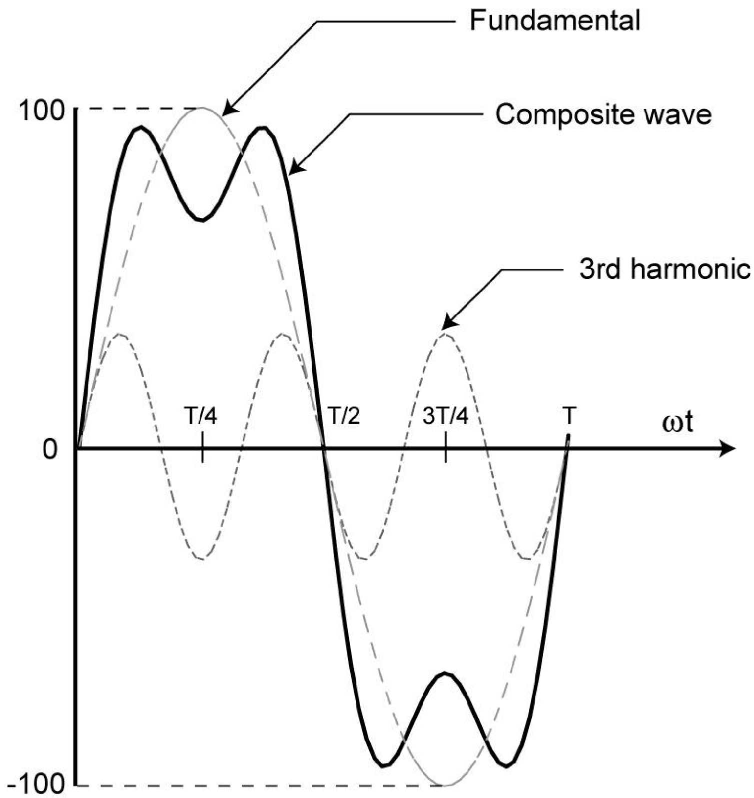
$y = A_o + (A_1 \sin\omega t + B_1 \cos\omega t) + (A_2 \sin 2\omega t + B_2 \cos 2\omega t) + (A_3 \sin 3\omega t + B_3 \cos 3\omega t) + (A_4 \sin 4\omega t + B_4 \cos 4\omega t) + \ldots\ldots\ldots (A_n \sin n\omega t + B_n \cos n\omega t)$

Where $\omega = 2\pi f$

As you will see, some complex waveforms will have a reduced expression of all sine or all cosine terms as below:

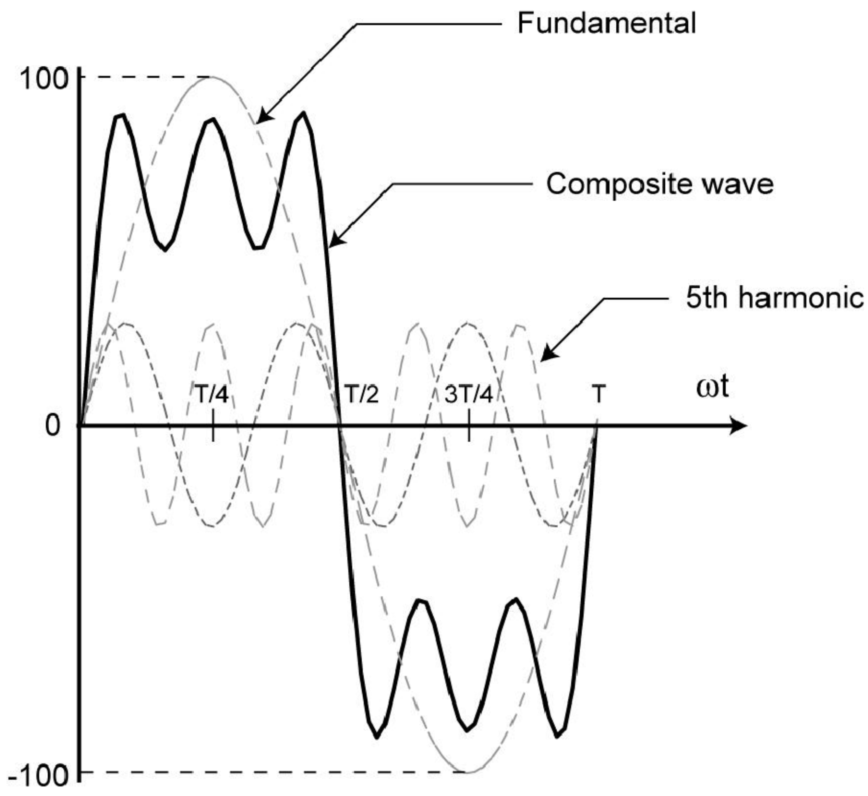
$y = A_o + A_1 \sin\omega t + A_2 \sin 2\omega t + A_3 \sin 3\omega t + A_4 \sin 4\omega t + \ldots . A_n \sin n\omega t$

Fourier Square Wave



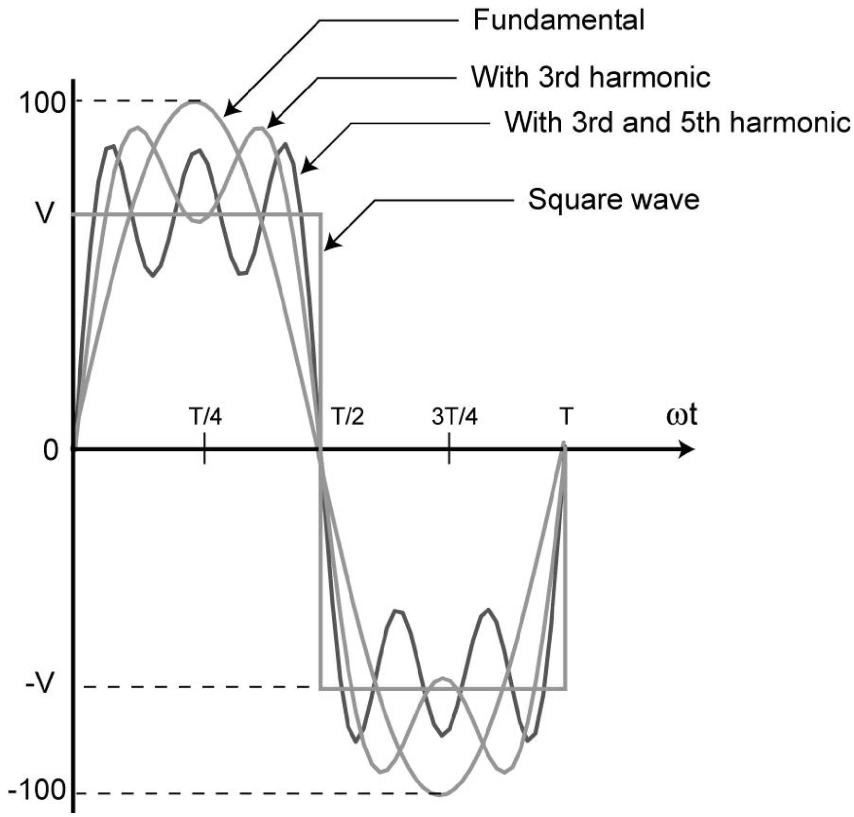
A discussion of this graphic is presented in the pages that follow. You can print this graphic for study purposes before going on.

Fourier Square Wave (cont.)



A discussion of this graphic is presented in the pages that follow. You can print this graphic for study purposes before going on.

Fourier Square Wave (cont.)



A discussion of this graphic is presented in the pages that follow. You can print this graphic for study purposes before going on.

Fourier Square Wave

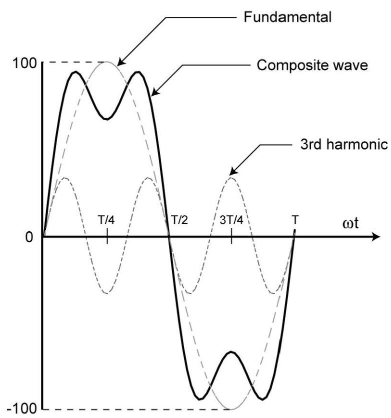
The non-sinusoidal waveform most used to illustrate the Fourier theory is the square wave. Recall that a square wave is a rectangular pulse train that has a 50% duty cycle. If put through a rigorous Fourier analysis, the square wave is shown to be made up of a fundamental sine wave and an infinite number of odd harmonics (3rd, 5th, etc.). The Fourier expression for a square wave is:

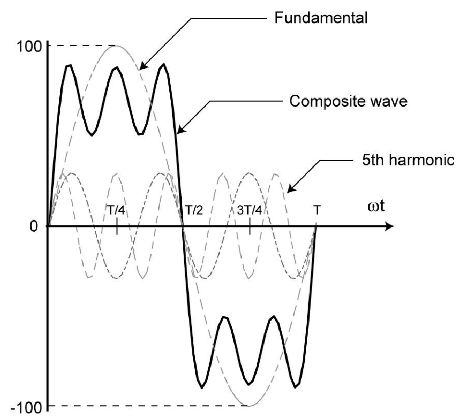
$$y = V_p + 4V_p/\pi\ [\sin\omega t + (\sin 3\omega t)/3 + (\sin 5\omega t)/5 + (\sin 7\omega t)/7 + \ldots.]$$

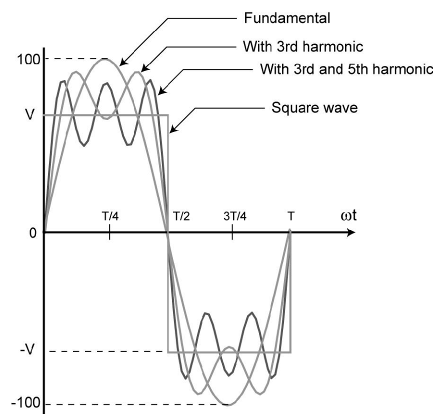
For an AC square wave centered on zero, there is no DC component so the first V_p term is not present. It would be present if the square wave switched between zero and V_p.

Fourier Square Wave (cont.)

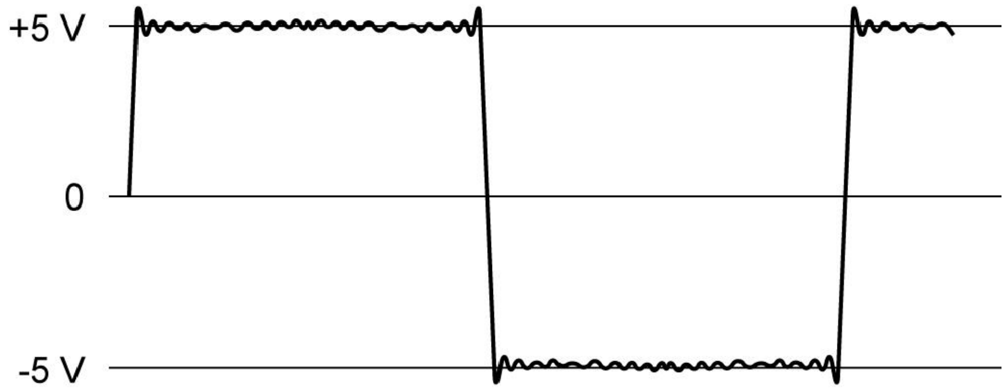
The Fourier expression states that if you take a sine wave at the same frequency as the square wave and add to it the odd harmonic sine waves at the correct amplitude, the result will be a square wave. The figures below illustrate this idea. Here only the third and fifth harmonics are added to the fundamental. The resulting composite waveform is beginning to approach a square wave shape. As more of the higher odd harmonics are added in, the sides of the wave become steeper and the top becomes flatter.







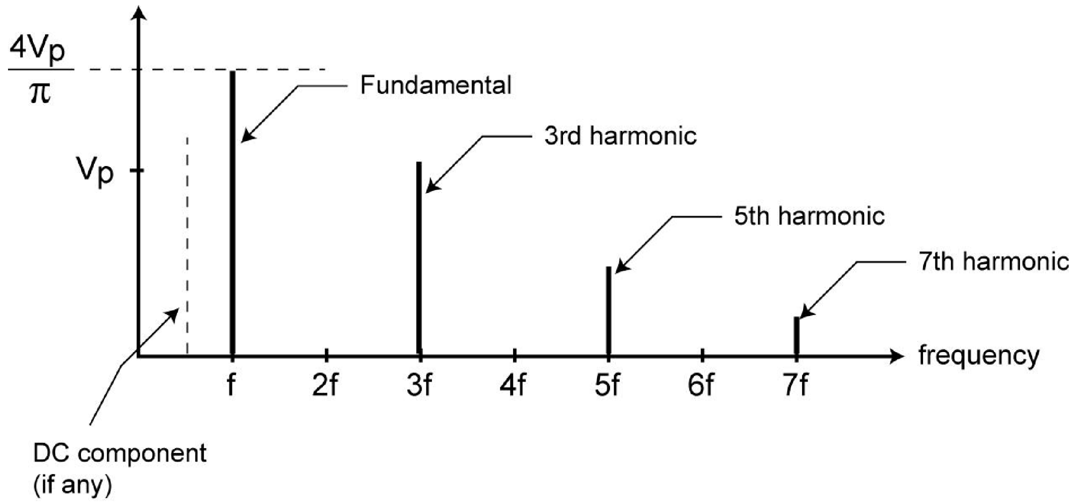
Fourier Square Wave (cont.)



A computer generated plot of a square wave made up of the fundamental plus the first 20 odd harmonics shows the resulting wave is nearly a perfect square wave.

This illustrates one very important point: you do not need an infinite number of harmonics to create the desired waveform.

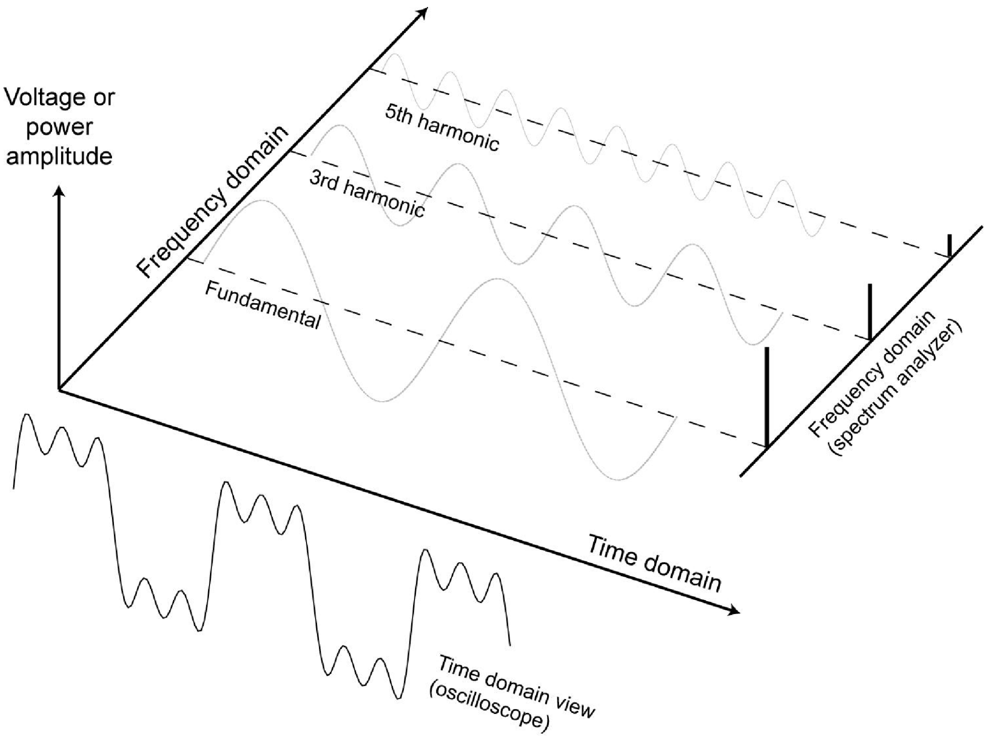
Frequency Domain Plot of the Square Wave



Fourier analysis makes it possible to think of a signal (a waveform) as combination of the fundamentals and harmonics of different amplitudes.

In the frequency domain plot of the square wave, the amplitudes of the harmonics decrease in proportion to their frequency. This was shown in the equation for the square wave given earlier. The DC component is shown dashed and may or may not be present. It is not present in this example.

Time and Frequency Domains of the Square Wave



In this figure, the relationship between the time and frequency domains is shown. Only the third and fifth harmonics are shown for simplicity.

DC Square Wave

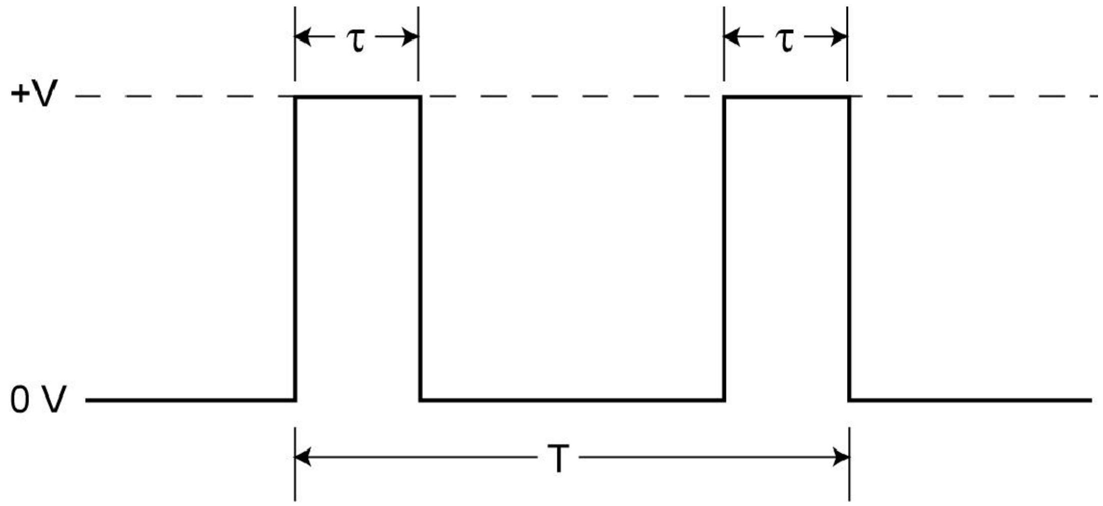
The square wave considered earlier was an AC wave with half the signal positive and the other half negative.

Many square waves are positive: that is, they switch from zero (or near zero) to some positive value. As a result, they have an average DC value.

The Fourier expression remains the same except for the term A_o at the beginning. This term is an expression of the DC average component of the wave.

You can view this signal as an AC square wave riding on, or superimposed, upon a DC level. The amplitudes of the fundamental and harmonics are the same as given earlier.

Rectangular Waves

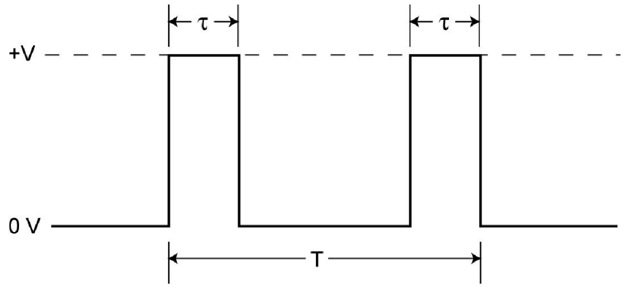


Duty cycle = $^{\tau}/_{T} \times 100$

$f = {}^{1}/_{T}$

A discussion of this graphic is presented in the pages that follow. You can print this graphic for study purposes before going on.

Rectangular Waves



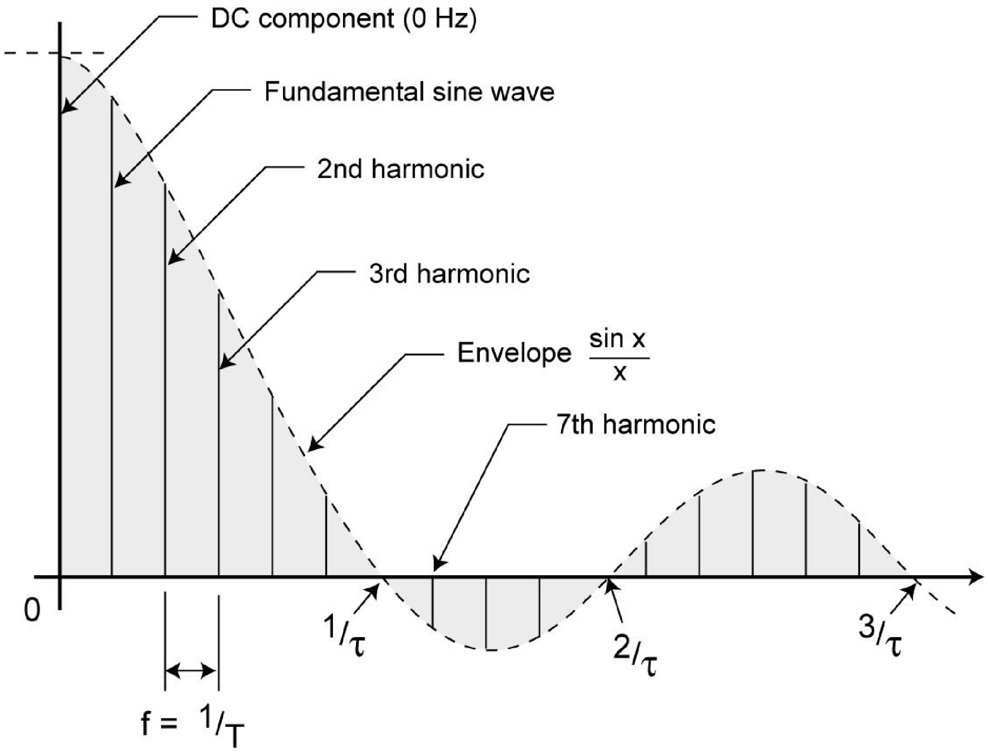
Duty cycle = $^{\tau}/_{T}$ x 100

f = $^{1}/_{T}$

A square wave is a rectangular wave with a 50% duty cycle. However, in most cases, rectangular waves do not have a 50% duty cycle. In some circuits, the pulse width is only a small percentage of the period. In other applications, a high duty cycle is used. Some applications actually have a varying duty cycle. Changing the pulse width and duty cycle of a rectangular wave changes the number and amplitude of the harmonics.

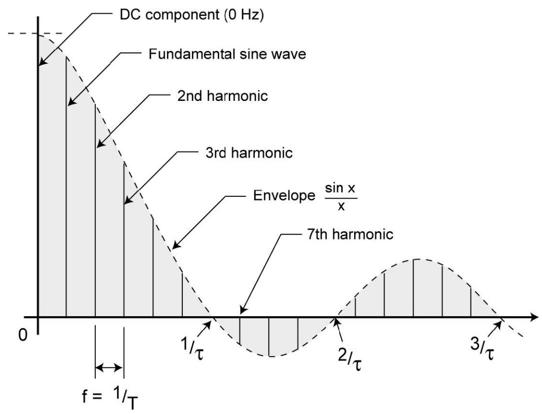
In the figure, the period is T and the pulse width is t while the duty cycle (DC) is t/T. (t is the lower case Greek tau.)

Frequency Domain of a Pulse Train



A discussion of this graphic is presented in the pages that follow. You can print this graphic for study purposes before going on.

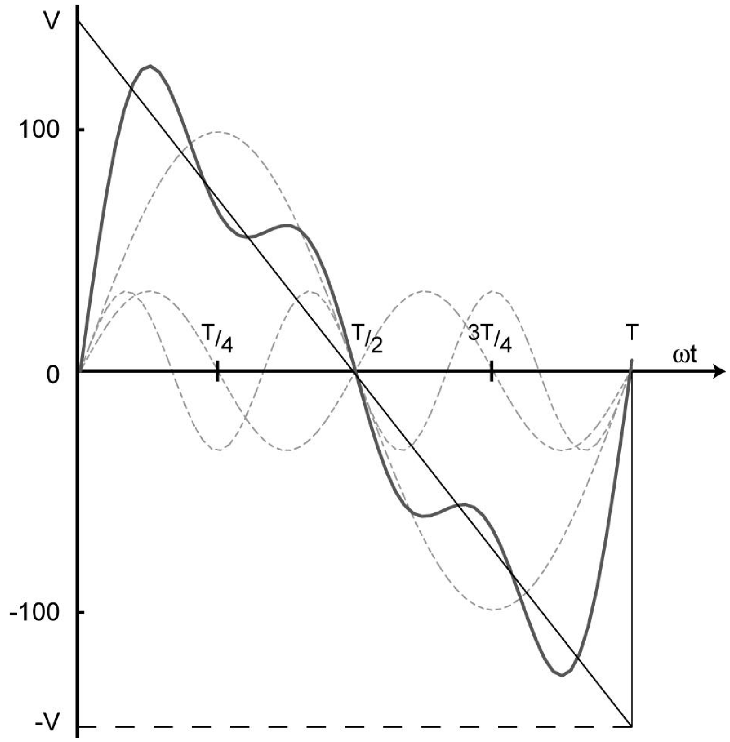
Frequency Domain of a Pulse Train



Harmonics are distributed in a repetitive pulse waveform. When the peaks of the fundamental and harmonic sine waves are connected to form a curve, it is called the envelope. The shape of the envelope is unique mathematically and is referred to as a (sin x)/x or sinc curve. Some of the higher harmonics lie below the frequency axis indicating a 180° phase shift.

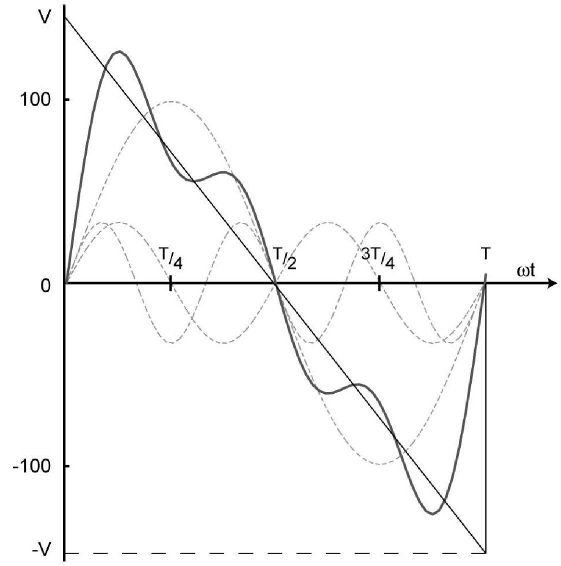
The key points on this curve are the places where the sinc curve crosses the frequency axis. These points are determined by the pulse width τ. The first crossing occurs at 1/τ, the second at 2/τ, the third at 3/τ and so on. At wide pulse widths, there are larger lower harmonics. As the pulse width narrows, the lower harmonics are diminished and a larger number of higher harmonics appear.

Fourier Sawtooth Wave



A discussion of this graphic is presented in the pages that follow. You can print this graphic for study purposes before going on.

Fourier Sawtooth Wave



This figure shows a sawtooth wave synthesized with sine waves. The fundamental, second, and third harmonics are added together to get a composite wave that begins to resemble the perfect sawtooth. Obviously, more higher harmonics need to be added to get a composite wave that better resembles a sawtooth.

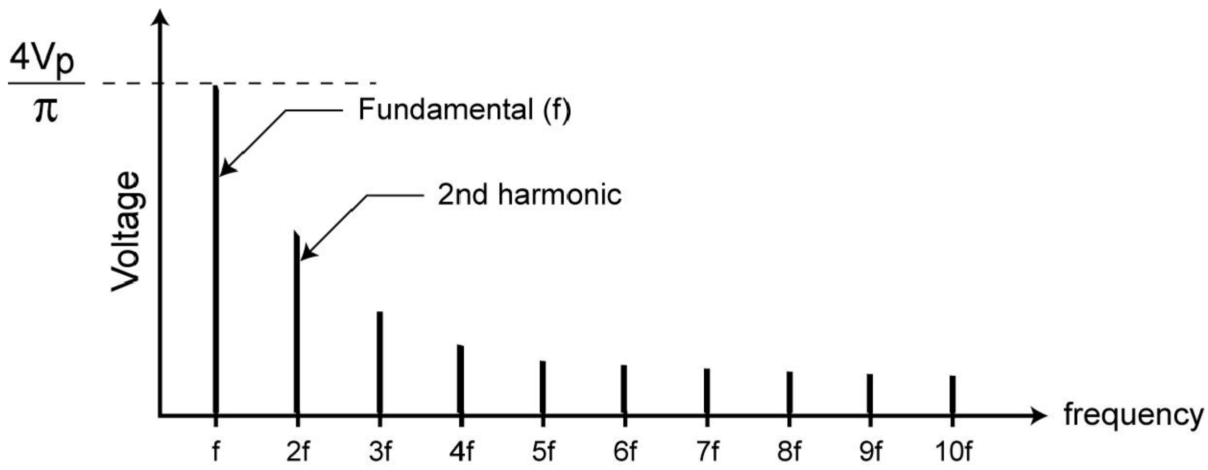
Fourier Sawtooth Wave (cont.)

The Fourier expression for the sawtooth is

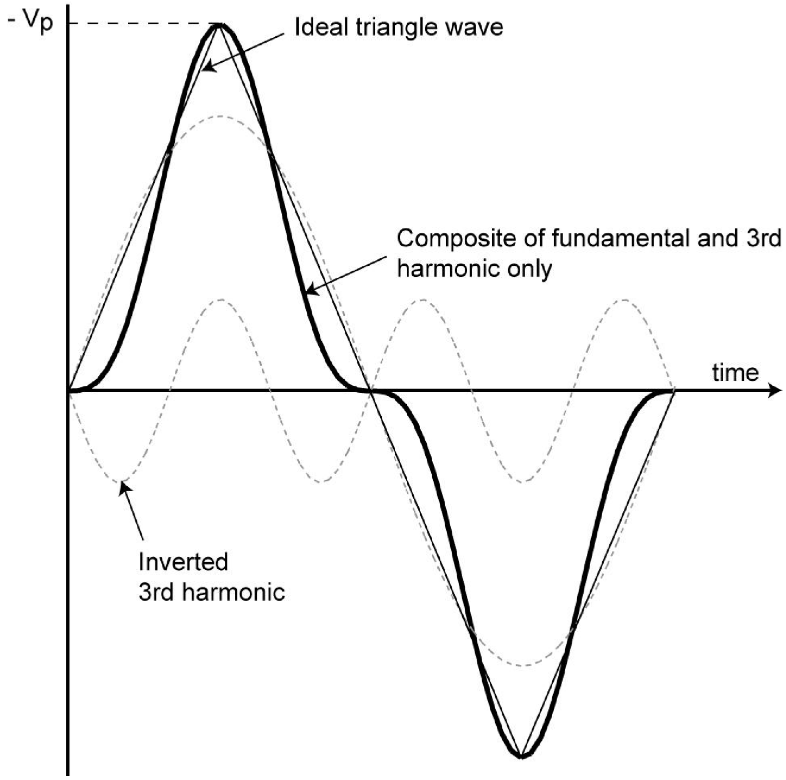
$y = 2V_p/\pi\ [\sin\omega t + (\sin 2\omega t)/2 + (\sin 3\omega t)/3 + (\sin 4\omega t)/4 + \ldots.]$

Both odd and even harmonics are contained in the sawtooth wave.

The figure below shows the frequency domain plot of the sawtooth.

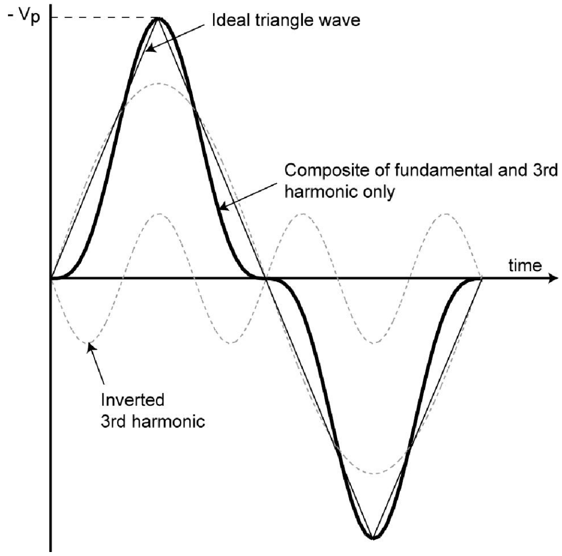


Fourier Triangle Wave



A discussion of this graphic is presented in the pages that follow. You can print this graphic for study purposes before going on.

Fourier Triangle Wave



This triangle wave contains only the odd harmonics as the Fourier expression below shows. But notice how the third harmonic does not go positive initially. It goes negative instead indicating a 180° phase shift.

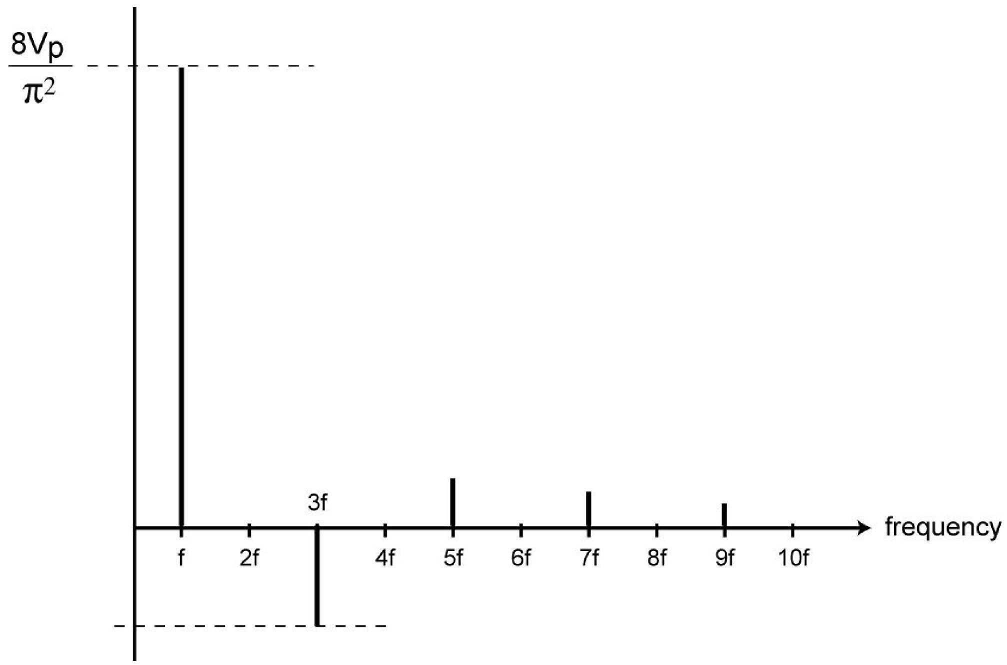
The Fourier expression is:

$$y = 8V_p/\pi^2 \,[\sin\omega t + (\sin 3\omega t)/9 + (\sin 5\omega t)/25 + (\sin 7\omega t)/49\ldots.]$$

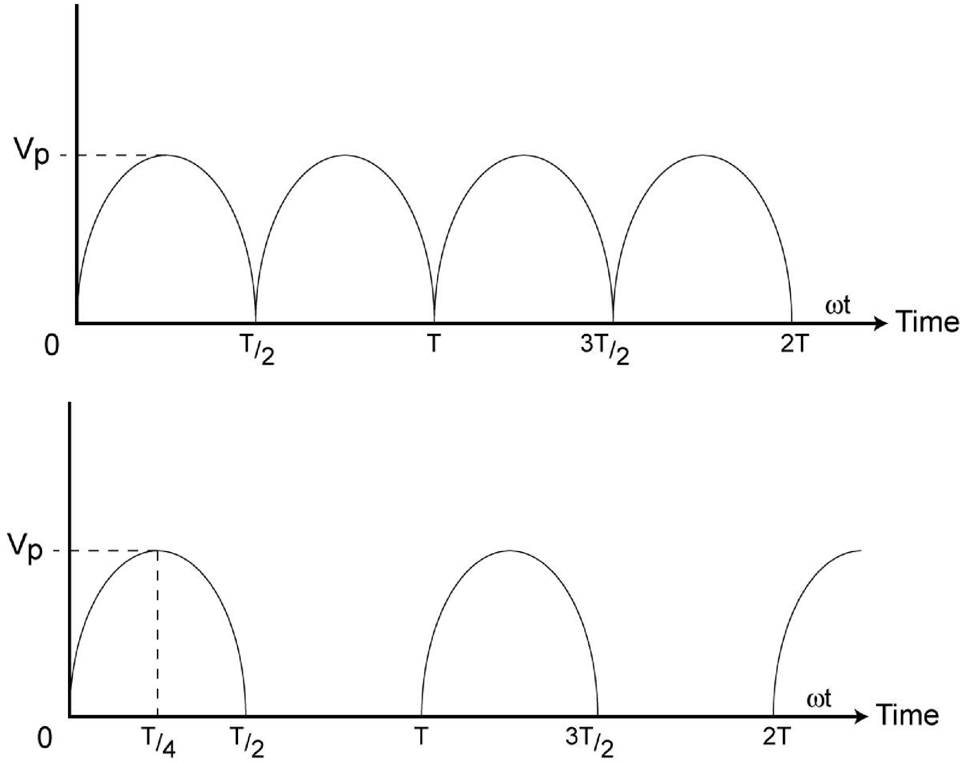
The amplitudes of the harmonics are a function of the square of the harmonic.

Frequency Domain Plot of the Triangle Wave

This figure shows the frequency domain plot of the triangle wave. The inverted 3rd harmonic is indicated in the frequency domain plot as a line below the horizontal frequency axis.



Full and Half Wave Rectified Sine Wave



A discussion of the graphic is presented in the pages that follow. You can print this graphic for study purposes before going on.

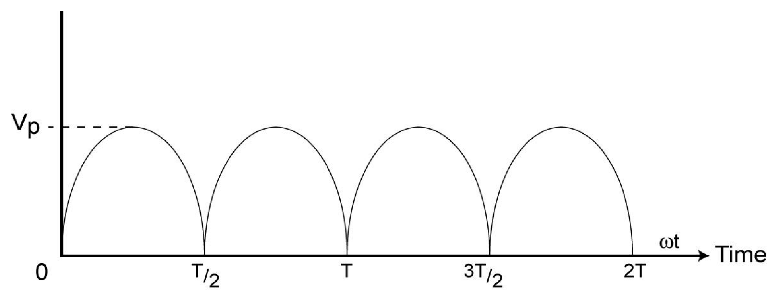
Full Wave Rectified Sine Wave

Rectified sine waves are also common in electronics. Rectification is the process of converting AC to DC using one or more diodes. The full wave rectified signal is shown below.

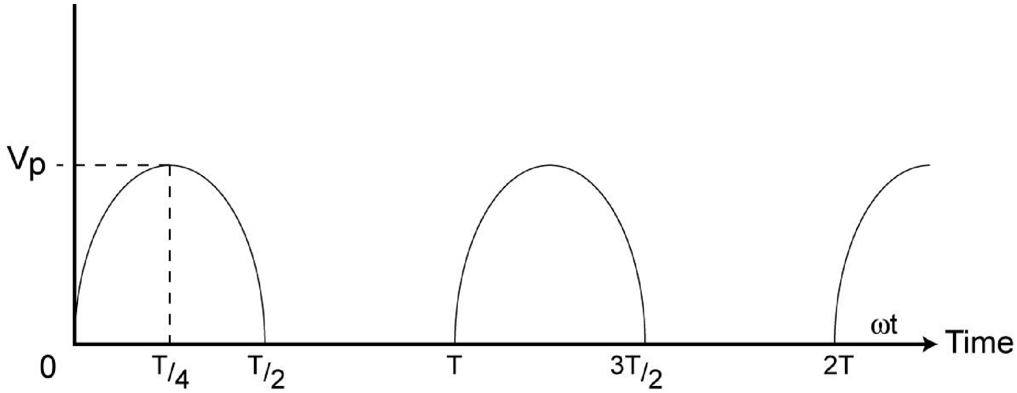
The Fourier expression shows that it has both odd and even harmonics.

$$y = 2V_p/\pi + 4V_p/\pi\ [(\cos\omega t)/3 - (\cos 2\omega t)/15 + (\cos 3\omega t)/35 + \ldots.)$$

The DC component is shown at the beginning of the expression.



Half Wave Rectified Sine Wave



The half wave rectified sine wave is shown above. The Fourier expression shows that it contains only even harmonics.

$y = V_p/\pi + V_p/2(\sin\omega t) - 2V_p/\pi\ [(\cos 2\omega t)/3 + (\cos 4\omega t)/15 + (\cos 6\omega t)/35\ldots.]$

The DC component is shown at the beginning of the expression.

Test your knowledge

Fourier Theory Knowledge Probe 3

The Fourier Theory

Click on Course Materials at the top of the page.

Then choose **Knowledge Probe 3**.