Fret Spacing Mathematics

This activity produces a guitar which has all of its frets located in the proper place along the neck such that the instrument plays in tune in all positions. Throughout history, specific spacing for each fret of a stringed instrument necessitates a precise measurement based on the exact scale length of the individual instrument. Pythagoras was the first known person to experiment with musical scalar intervals near 500 BC. Later, in the 16th century, Galileo's father, Vincenzo Galilei, was credited with developing the "Rule of 18," used for centuries by instrument makers to determine the fret scale length of their instruments. For any given vibrating string length they would simply divide the length of the string by 18, yielding the distance from the nut to the first fret. By subtracting that figure from the original string length they arrived at a new shorter scale measurement, which was then divided once again by 18 and resulted in the distance between the first and second frets. They continued in this manner until the entire scale was determined. Over the years the constant divisor 18 has been refined to 17.817 (derived from a formula based on $2^{1/12}$, or the 12th root of 2) resulting in more accurate scales. This activity is suitable for grades 9-12.

Learning Objectives:

1. Students will apply mathematical concepts of stringed instrument scale length by interpreting, explaining, solving, and using an equation for calculating fret spacing.
2. Students will determine accurate fret locations by calculating the exact fret locations for the first five fret positions when given a specific scale length of a stringed instrument.

Standards:

CCSS.Math.Content.HSA-SSE.A.1b Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P*.

CCSS.Math.Content.HSA-SSE.B.3c Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%*.

CCSS.Math.Content.HSA-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution.

CCSS.Math.Content.HSA-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

Materials Required:

1. Paper, pencil, calculator

Safety:

N/A

References:

Donald Duck in Mathmagicland. (1959). [You Tube video excerpt]. Disney Motion Pictures. Retrieved from http://youtu.be/iEVGQKwKeCc

French, Mark. (2012). *Math and music.* [You Tube video]. Retrieved from http://youtu.be/0raHDoTIGps

French, Mark. (2012). *Music and scales.* Brain Waves.avi. [You Tube video]. Retrieved from http://youtu.be/AcFEoFvLEWU

Huvard, A. J. (1997). *Formula for computing a fret scale.* Retrieved 5 January 2014 from http://www.cybozone.com/luthier/instruments/fretscale.html

Passy. (2012). *Guitar mathematics.* [Website]. Retrieved from http://passyworldofmathematics.com/guitar-mathematics/

Stewart-Macdonald. (2014). *Scale length explained.* Retrieved 5 January 2014 from http://www.stewmac.com/freeinfo/Fretting/a-scalelength.html

Activity:

·**Play YouTube Video:** Excerpt Donald Duck in Mathmagicland: http://youtu.be/iEVGQKwKeCc

Play YouTube Video: Dr. Mark French, Math and Music: http://youtu.be/0raHDoTIGps

This video shows a relationship between math and music. Dr. French demonstrates how to find the 12^{th}-root of 2 and how it applies to a standard music scale. This number is much more accurate in calculating fret location on a guitar than the "Rule of 18." Since this method is based on an international standard, every stringed instrument fretboard layout can be calculated accurately.

You can use this knowledge to calculate frequencies of notes along a scale or to calculate the spacing between the frets of a guitar. The relationship of Dr. French's discussion on frequencies and our calculation of fret spacing is based on this mathematical equation and the need to control frequencies to play specific musical notes on a guitar.

The mathematical formula to calculate fret spacing using the 12^{th}-root of 2 *(*The equal-tempered chromatic scale) as the constant:

$$X_n = L\left(1 - \frac{1}{r^n}\right)$$

Or, written on one line: $X_n = L\,[1 - (1 \div r^n)]$

Where: X_n is the distance of the fret n to the string nut at the end of the fretboard

n is the current fret being calculated

L is the scale length

$r = 2^{(1/12)}$ = 1.059463

NOTE: To derive the constant value of 17.817 from the "12th-root of 2" formula, use these steps:

$$\text{Constant} = \frac{\sqrt[12]{2}}{\left(\sqrt[12]{2}\right) - 1}$$

$$\text{Constant} = \frac{1.059463}{1.059463 - 1}$$

Constant = 17.817

Rule of 18 Fret Spacing Relationships

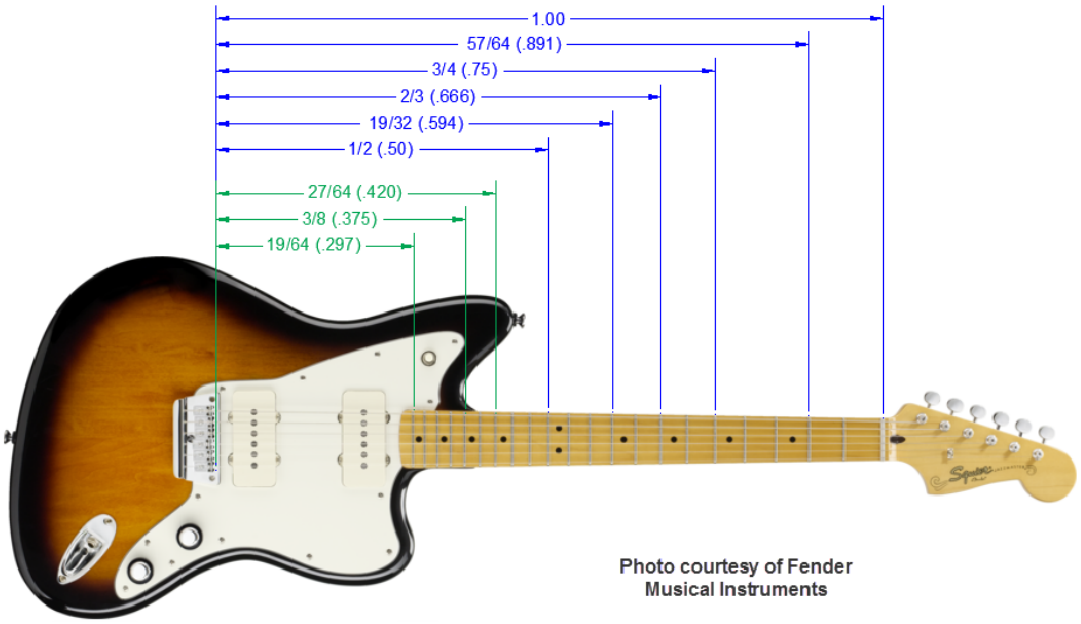
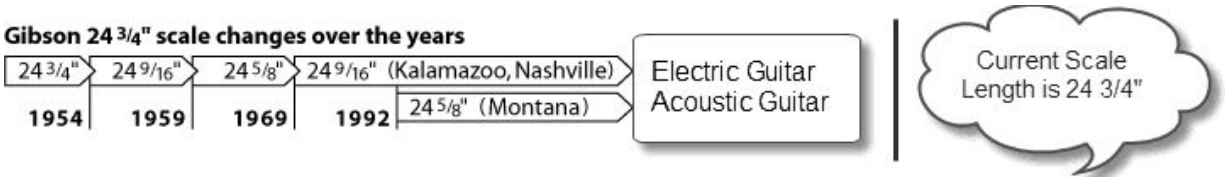


Photo courtesy of Fender Musical Instruments

Exercises #1-5: Fender™
One of the most common scale lengths is the Fender 25-1/2 "guitar scale. Found on Stratocasters®, Telecasters®, and the huge variety of instruments inspired by them as well as the replacement, and custom parts available for them. The 25-1/2" produces a rich, strong, bell-like tone, and defined low-end.

**Activity A: Determine the first 5 fret locations for a guitar with a standard 25.5-inch scale length*

Exercises #6-10: Gibson™
The Gibson 24-3/4" scale is also very common, but it is also the most confusing of all scale lengths—this is because Gibson has changed the scale length standard over the years. It rarely ever measures out to be exactly 24-3/4 inches! This scale has gradually changed over the past fifty or so years due to changes in production equipment. The approximation that is currently used is 24 ¾"



Being shorter than the Fender™ 25-1/2" scale, the Gibson™ 24-3/4" scale has a lower tension and what some might describe as a "warmer" tone.

**Activity B: Determine the first 5 fret locations for a guitar with a 24.625 scale length.*

Formula for Calculating Fret Spacing:

$$D_n = [(L - D_{n-1}) \div 17.817] + D_{n-1}$$

Where:
L = Scale Length
n = Fret Position
D_n = Distance from Nut to Current Fret Position (rounded to thousandths place)
D_{n-1} = Distance from Nut to Previous Fret Position (round to thousandths place)

17.817 = constant for calculating fret positions $= \frac{\sqrt[12]{2}}{\left(\sqrt[12]{2}\right)-1}$

Show all procedural steps: formula, substitution, calculation, and answer

(See following page for guided practice on how to calculate using this formula)

Guided Practice:

Determine the first 3 fret locations for a stringed instrument with a 24" scale length.

Formula: $D_n = [(L - D_{n-1}) \div 17.817] + D_{n-1}$

Where:
L = Scale Length
n = Fret Position
D_n = Distance from Nut to Current Fret Position (round to thousandths place)
D_{n-1} = Distance from Nut to Previous Fret Position (round to thousandths place)

17.817 = constant for calculating fret positions $= \frac{\sqrt[12]{2}}{\left(\sqrt[12]{2}\right)-1}$

Fret Spacing Example Calculation for Fret Position #1

L = 24" D_1 = unknown variable D_0 = 0" (Position of the Nut)

$D_1 = [(L - D_0) \div 17.817] + D_0$
$= [(24 + 0) \div 17.817] + 0$
$= [24 \div 17.817] \approx 1.347$"

Fret Position #2

L = 24" D_2 = unknown variable D_1 = 1.347"

$D_2 = [(L - D_1) \div 17.817] + D_1$
$= [(24 - 1.347) \div 17.817] + 1.347$
$= [22.653 \div 17.817] + 1.347$
$= 1.271426166 + 1.347 \approx 2.618$"

Fret Position #3

L = 24" D_3 = unknown variable D_2 = 2.618"

$D_3 = [(L - D_2) \div 17.817] + D_2$
$= [(24 - 2.618) \div 17.817] + 2.618$
$= [21.382 \div 17.817] + 2.618$
$= 1.200089802 + 2.618 \approx 3.818$"

Practice Activity Answer Sheet

Formula: $D_n = [(L - D_{n-1}) \div 17.817] + D_{n-1}$

Exercise #1: Fender™ *L* = 25.5", Fret Position #1

L = 25.5" D_1 = unknown variable D_0 = 0" (Position of the Nut)

$D_1 = [(L - D_0) \div 17.817] + D_0$
$= [(25.5 + 0) \div 17.817] + 0$
$= [25.5 \div 17.817] \approx 1.431"$

Exercise #2: Fender™ *L* = 25.5", Fret Position #2

L = 25.5" D_2 = unknown variable D_1 = 1.431"

$D_2 = [(L - D_1) \div 17.817] + D_1$
$= [(25.5 - 1.431) \div 17.817] + 1.431$
$= [24.069 \div 17.817] + 1.431$
$= 1.350900825 + 1.431 \approx 2.782"$

Exercise #3: Fender™ *L* = 25.5", Fret Position #3

L = 25.5" D_3 = unknown variable D_2 = 2.782"

$D_3 = [(L - D_2) \div 17.817] + D_2$
$= [(25.5 - 2.782) \div 17.817] + 2.782$
$= [22.718 \div 17.817] + 2.782$
$= 1.275074367 + 2.782 \approx 4.057"$

Exercise #4: Fender™ *L* = 25.5", Fret Position #4 ≈ 5.261"
Exercise #5: Fender™ *L* = 25.5", Fret Position #5 ≈ 6.397"

Continue procedure as noted above for Exercises #5 – 10

Exercise #6: Gibson™ *L* = 24.625", Fret Position #1 ≈ 1.382"
Exercise #7: Gibson™ *L* = 24.625", Fret Position #2 ≈ 2.687"
Exercise #8: Gibson™ *L* = 24.625", Fret Position #3 ≈ 3.918"
Exercise #9: Gibson™ *L* = 24.625", Fret Position #4 ≈ 5.080"
Exercise #10: Gibson™ *L* = 24.625", Fret Position #5 ≈ 6.177"

Advanced examples using Dr. French's 12th-root of 2 formula

Determine the first 3 fret locations for a stringed instrument with a 24" scale length.

Fret Spacing Example calculation

Fret position #1 of a 24" scale length guitar

$$X_n = L\,[1 - (1 \div r^n)]$$

X_n = 24 [1 – (1/1.059463^1)]
X_n = 24 [1 – (1/1.059463)]
X_n = 24 [1 – 0.943714]
X_n = 24 [0.056286]
X_n = 1.350"

X_n is the distance of the fret n to the string nut at the end of the fretboard

n is the current fret being calculated

L is the scale length

= 1.059463

Fret position #2

$$X_n = L\,[1 - (1 \div r^n)]$$

X_n = 24 [1 – (1/1.059463^2)]
X_n = 24 [1 – (1/1.122843)]
X_n = 24 [1 – 0.890596]
X_n = 24 [0.109404]
X_n = 2.625"

Fret position #3

$$X_n = L\,[1 - (1 \div r^n)]$$

X_n = 24 [1 – (1/1.059463^3)]
X_n = 24 [1 – (1/1.18981)]
X_n = 24 [1 – 0.840468]
X_n = 24 [0.159532]
X_n = 3.828"

Advanced Practice:

Problem 1

Daisy Rock® Guitars creates guitars for Teenage and Pre- Teen girls. Their design allows the girls reach the frets easier with a shorter scale length. Determine the first 4 fret locations for a stringed instrument with a 22.5 inch scale length used on a Daisy Rock® guitar.

Fret 1:
Fret 2:
Fret 3:
Fret 4:

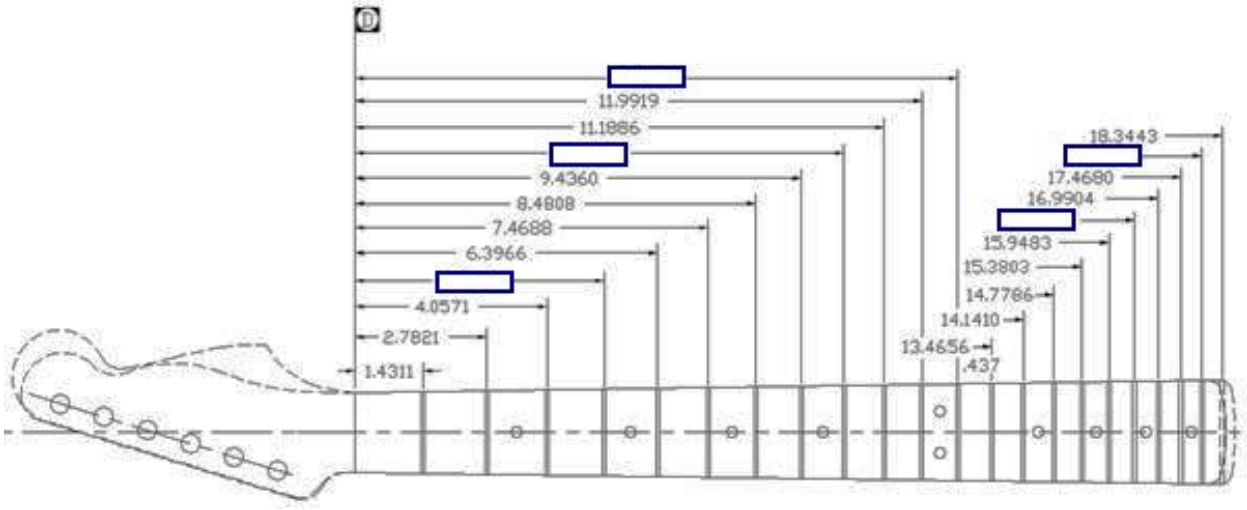
Problem 2

Determine the Fret location 12, 15, 17, 19 for a Bass Guitar with a 34 inch scale length.

Fret 12:
Fret 15:
Fret 17:
Fret 19:

You must show all your work neatly on a separate sheet, including formula, substitution, calculation, and answer.

Problem 3



Fill in the missing fret positions as indicated on the neck. Use the existing fret spacing numbers to help determine the distance.

Hint: use the equation to find the overall scale length first.

Advanced Practice Activity Answers

Problem 1: Solution for Fret Spacing Calculation **for 22.5" scale length**

Fret 1 = 22.5 [1 – (1/1.059463^{1})] = 22.5 [1 – (1/1.059)] = 22.5 [1 – 0.944] = 22.5 [0.056] **= 1.253"**	*Fret 2* = 22.5 [1 – (1/1.059463^{2})] = 22.5 [1 – 1/1.122] = 22.5 [1 – 0.891] = 22.5 [0.109] **= 2.453"**
Fret 3 = 22.5 [1 – (1/1.059463^{3}] = 22.5 [1 – (1/1.189)] = 22.5 [1 – 0.841] = 22.5 [0.159] **= 3.578"**	*Fret 4* = 22.5 [1 – (1 / 1.059463^{4})] = 22.5 [1 – (1 / 1.260)] = 22.5 [1 – 0.794] = 22.5 [0.206] **= 4.635"**

Problem 2: Solution for Fret Spacing Calculation **for 34" scale length problem**

Fret 12 = 34 [1 – (1/1.059643^{12})] = 34 [1 – (1/2.00407)] = 34 [1 – 0.49898] = 34 [0.50101] **= 17.034"**	Fret 15 = 34 [1 – (1/1.059643^{15})] = 34 [1 – (1/2.38447)] = 34 [1 – 0.41938] = 34 [0.58062] **= 19.741"**
Fret 17 = 34 [1 – (1/1.059643^{17})] = 34 [1 – (1/2.67739)] = 34 [1 – 0.37349] = 34 [0.62650] **= 21.301"**	Fret 19 = 34 [1 – (1/1.059643^{19})] = 34 [1 – (1/3.0063)] = 34 [1 – 0.33263] = 34 [0.66736] **= 22.690"**

Problem 3: Solutions for Fret Spacing Calculation **neck graphic for frets 4, 9, 12, 18, 21**

Calculating scale length of guitar neck : $X_n = L\,[1 - (1 / r^n)]$
1.4311= L [1- (1/1.059643^1)] (Divide both sides by constant)
1.4311/[1-(1/1.059643^1)] = L
L=1.4311/[1- (1/1.059643^1)]
L=1.4311/(1-.943714)
L=1.4311/0.056285
L= 25.42"

Fret 4

= 25.42 [1 – (1/1.059643^4)]
= 25.42 [1 – (1/1.335)]
= 25.42 [1 – 0.749]
= 25.42 [0.251]
= 6.380"

Fret 9

= 25.42 [1 – (1/1.059643^9)]
= 25.42 (1 – (1/1.414)]
= 25.42 [1 – 0.707]
= 25.42 [0.293]
= 7.448"

Fret 12

= 25.42 [1 – (1/1.059643^{12})]
= 25.42 [1 – (1/2)]
= 25.42 [1 – 0.5]
= 25.42 [0.5]
= 12.710"

Fret 18

= 25.42 [1 – (1/1.059643^{18})]
= 25.42 [1 – (1/2.828)]
= 25.42 [1 – 0.354]
= 25.42 [0.646]
= 16.421"

Fret 21

= 25.42 [1 – (1/1.059643^{21})]
= 25.42 [1 – (1/3.364)]
= 25.42 [1 – 0.297]
= 25.42 [0.703]
= 17.870"

Name _____

Assessment
Fret Spacing Mathematics

1. Who is the first known person to experiment with determining scalar intervals for stringed instruments (Scale Lengths)?

A. Pythagoras
B. Galileo
C. Les Paul
D. Eddie Van Halen
E. None of these

2. Vincenzo Galilei was credited with developing the "Rule of 18" in the 16th century.

True -or- False

3. The formula for calculating fret spacing is derived from the "Rule of 18" and which of the following?

A. The quadratic formula
B. The twelfth root of 2
C. The Pythagorean theorem
D. Pi
E. All of these

4. A guitar's scale length can be calculated by measuring the distance from the front edge of the nut to the center of the 12th fret, then doubling that measurement.

True -or- False

5. One of the most common scale lengths for electric guitars is Fender's 25-1/2" scale.

True -or- False

6. Match each symbol in the fret calculation formula $D_n = [(L - D_{n-1}) \div 17.817] + D_{n-1}$ with what it represents from the options below labeled a through e.

_____	L	a. Constant for calculating fret position
_____	n	b. Distance from nut to previous fret position
_____	D_n	c. Distance from nut to current fret position
_____	D_{n-1}	d. Scale length
_____	17.817	e. Fret position

7. A guitar with a shorter scale length has a lower string tension than a guitar with a longer scale length if both are tuned to the same pitch.

True -or- False

8. To implement the “Rule of 18,” 16th century instrument makers would begin with which procedure?

A. Divide the string length by 2 (18 times)
B. Subtract 18 from the string length
C. Divide the string length by 2 to the 18th power
D. Divide the string length by 18
E. The "Rule of 18" is just an expression and was never actually used to calculate fret spacing.

9. Select the correct equation to locate the 2nd fret on the neck of a 13.5" scale length mandolin.

Formula: Formula: $D_n = [(L - D_{n-1}) \div 17.817] + D_{n-1}$

A. $D_2 = [(25.5 - 0) \div 17.817] + 0$
B. $D_2 = [(13.5 - .76) \div 18] + 25.5$
C. $D_2 = [(13.5 - .76) \div 17.817] + 13.5$
D. $D_2 = [(13.5 -.76) \div 17.817] + .76$
E. $D_2 = [(13 - .76) \div 18 + 0$

10. The Gibson 24-3/4" scale length rarely measures out to be 24-3/4" because the scale length has gradually changed over the past 50+ years due to changes in production equipment.

True -or- False

Assessment Key:

1. A - Pythagoras
2. True
3. B - The twelfth root of 2
4. True
5. True
6. d. Scale length
   e. Fret position
   c. Distance from nut to current fret position
   b. Distance from nut to previous fret position
   a. Constant for calculating fret position
7. True
8. D - Divide the string length by 18
9. D - $D_2 = [(13.5 - .76) \div 17.817] + .76$
10. True

Reviewing Faculty Cohort Members:

Chad McCormack, Wells High School, Wells, ME (3/18)
Dave Parker, Noble High School, North Berwick, ME (3/18)